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Further Solutions in Streamwise Corner Flow with Wall Suction

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Introduction

In Ref. 1, Barclay and El-Gamal have given solutions for the laminar boundary-layer flow along a rectangular streamwise corner when the flow is subjected to a uniform suction at the walls. Here the boundary-layer problem is reconsidered for the somewhat simpler case where the suction is proportional to the square root of the reciprocal of the local Reynolds number R_x. Introducing $R_x \equiv 2U_x/\nu$ but otherwise adopting the same notation as in Ref. 1, we therefore replace the suction velocity $v_s(x)=$ const with the condition that $v_s(x) = -\epsilon U R_x^{-1/2}$, $\epsilon=$const. Noting that σ (Ref. 1) is now independent of x (in fact, $\sigma=\epsilon$) the boundary-layer equations and boundary conditions are obtained directly from Eqs. (2-7) in Ref. 1 to be

$$-(\eta u^* - v^*)\frac{\partial u^*}{\partial \eta} - (\zeta u^* - w^*)\frac{\partial u^*}{\partial \zeta} = \nabla^2 u^* \tag{1a}$$

$$-u^*\left(v^* + \eta\frac{\partial v^*}{\partial \eta} + \zeta\frac{\partial v^*}{\partial \zeta}\right) + v^*\frac{\partial v^*}{\partial \eta} + w^*\frac{\partial v^*}{\partial \zeta} = \nabla^2 v^* - \frac{\partial p^*}{\partial \eta} \tag{1b}$$

$$-u^*\left(w^* + \eta\frac{\partial w^*}{\partial \eta} + \zeta\frac{\partial w^*}{\partial \zeta}\right) + v^*\frac{\partial w^*}{\partial \eta} + w^*\frac{\partial w^*}{\partial \zeta} = \nabla^2 w^* - \frac{\partial p^*}{\partial \zeta} \tag{1c}$$

$$-\eta\frac{\partial u^*}{\partial \eta} - \zeta\frac{\partial u^*}{\partial \zeta} + \frac{\partial v^*}{\partial \eta} + \frac{\partial w^*}{\partial \zeta} = 0 \tag{1d}$$

$$\begin{aligned}
&\eta=0; && u^*=0, && v^*=\epsilon v_c^*(\zeta), && w^*=0\\
&\zeta=0; && u^*=0, && v^*=0, && w^*=\epsilon v_c^*(\eta)\\
&\eta\to\infty; && u^*=\bar u(\zeta), && v^*=\bar w(\zeta), && w^*=\bar v(\zeta)\\
&\zeta\to\infty; && u^*=\bar u(\eta), && v^*=\bar v(\eta), && w^*=\bar w(\eta)
\end{aligned} \tag{2}$$

where $\bar u(\eta), \bar v(\eta), \bar w(\eta)$ are the solutions of

$$-(\eta\bar u - \bar v)\frac{\partial \bar u}{\partial \eta} = \frac{\partial^2 \bar u}{\partial \eta^2} \tag{3}$$

$$-(\eta\bar u - \bar v)\frac{\partial \bar w}{\partial \eta} - \bar u\bar w = -\lim_{\eta\to\infty}\bar v + \frac{\partial^2 \bar w}{\partial \eta^2} \tag{4}$$

$$-\eta\frac{\partial \bar u}{\partial \eta} + \frac{\partial \bar v}{\partial \eta} = 0 \tag{5}$$

subject to

$$\bar u(0)=0, \qquad \bar u(\infty)=1, \qquad \bar v(0)=-\epsilon \tag{6a}$$

$$\bar w(0)=0, \qquad \bar w(\infty)=\bar v(\infty) \tag{6b}$$

As before, $v_c^*(t)$ is considered to be indeterminate and a solution to Eq. (1) must involve an assumption for the form of $v_c^*(t)$ satisfying the conditions $v_c^*(0)=0$ and $v_c^*(\infty)=-1$.

Solution at the Side Edge ($\zeta\to\infty$)

Equations (1) and (3-5) above are the same as for zero suction, but their solutions are parametrically dependent on the proportionality constant ϵ appearing in the boundary conditions. The zero suction case $\epsilon=0$ has been solved by Rubin[2] and Rubin and Grossman.[3] Equations (3) and (5), with Eq. (6a), are uncoupled from Eqs. (4) and (6b) and have been solved for $\bar u$ and $\bar v$ by Schlichting and Bussmann.[4] Improved numerical results have been given by Emmons and Leigh.[5]

Letting $\bar u = F'(\eta)$, $\bar w = \beta H'(\eta)$ where $\beta = \lim_{\eta\to\infty}\bar v$, and substituting in Eqs. (4) and (6b) we obtain Rubin's equations

$$H''' + (H'F)' = 1, \qquad H'(0)=0, \qquad H'(\infty)=1$$

for which the solution is

$$H'(\eta) = F''(\eta)\int_0^\eta \frac{(t-\beta)}{\ddot F(t)}\,\mathrm{d}t \tag{7}$$

$\bar w(\eta)$ is shown in Fig. 1. The outflow β from the boundary layer is zero at $\epsilon=0.731$ (approx.) and $\bar w$ is therefore zero everywhere at the side edge for $\epsilon=0.731$. This interesting result separates the cross flow into two types, one displaying a region of flow reversal ($0\le\epsilon<0.731$) and one in which $\bar w(\eta)$ is everywhere negative ($\epsilon>0.731$).

When $\epsilon\sim4$ or larger, all of the velocity components have practically reached asymptotic forms valid for large ϵ. It has been shown by Watson[6] in his analysis for two-dimensional flows that the asymptotic solutions for $\bar u$ and $\bar v$ take the forms

$$\bar u = 1 - e^{-\epsilon\eta}, \qquad \bar v = -\epsilon \tag{8}$$

It may also be shown that the asymptotic solution for $\bar w$ is

$$\bar w = -\epsilon(1 - e^{-\epsilon\eta}) = -\epsilon\bar u \tag{9}$$

Received Nov. 16, 1982; revision received July 8, 1983.

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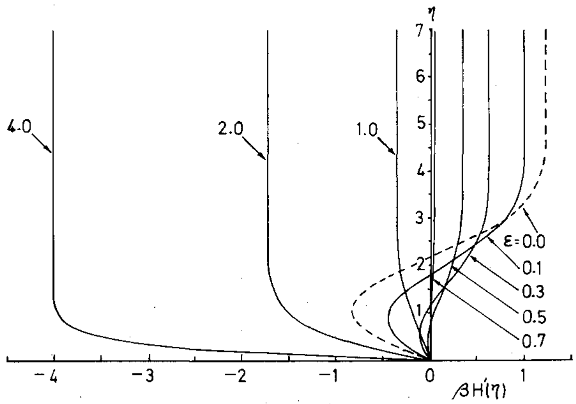

Fig. 1 Dimensionless cross flow velocity component $\bar{w} = \beta H'(\eta)$.

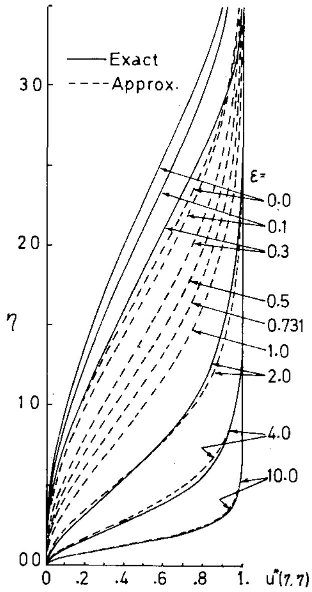

Fig. 2 Comparison of exact solutions for $u^*(\eta,\eta)$ for small and large ϵ and the approximate solution.

Small ϵ Solution

Paralleling the method used in Ref. 1, Eqs. (1) are solved for small and large values of ϵ by means of expansions for the dependent variables, even although this requires re-solving Eqs. (3-5) in the same manner for consistency. Thus, for small ϵ we start by assuming that

$$\bar{u}(\eta;\epsilon) = \sum_{n=0}^{\infty} \bar{u}_n(\eta)\epsilon^n, \qquad \bar{v}(\eta;\epsilon) = \sum_{n=0}^{\infty} \bar{v}_n(\eta)\epsilon^n$$

$$\bar{w}(\eta;\epsilon) = \sum_{n=0}^{\infty} \bar{w}_n(\eta)\epsilon^n$$

Substitution in Eqs. (3), (5), and (6a) gives a system of equations already solved for $\bar{u}_0$, $\bar{v}_0$ (Ref. 7), $\bar{u}_1$, $\bar{v}_1$, (Ref. 8), and $\bar{w}_0$ (Ref. 2). To complete the solution to order $n=1$ we can show that $\bar{w}_1 = \beta_1 H_1'(\eta)$ where $\beta_1 = \lim_{\eta\to\infty} \bar{v}_1 = -2.05418$ and

$$H_1'(\eta) = \frac{1}{\beta_1} f''(\eta) \int_0^{\eta} \frac{-f_1(t)\beta\dot{H}(t) + \beta_1 t - 2\beta\beta_1}{\ddot{f}(t)} \mathrm{d}t \qquad (10)$$

in which $f'(\eta) = \bar{u}_0$, $f_1'(\eta) = \bar{u}_1$, and $\eta f_1' - f_1 = \bar{v}_1$.

Substituting corresponding series for $u^*(\eta,\zeta;\epsilon)$, $v^*(\eta,\zeta;\epsilon)$, $w^*(\eta,\zeta;\epsilon)$, and $p^*(\eta,\zeta;\epsilon)$ in Eqs. (1) and retaining only terms of order $n=1$ gives

$$-(\eta u_0^* - v_0^*)\frac{\partial u_1^*}{\partial\eta} - (\zeta u_0^* - w_0^*)\frac{\partial u_1^*}{\partial\zeta} - \frac{\partial u_0^*}{\partial\eta}(\eta u_1^* - v_1^*) - \frac{\partial u_0^*}{\partial\zeta}(\eta u_1^* - w_1^*) = \nabla^2 u_1^*$$

$$-u_0^* v_1^* - v_0 u_1^* - \frac{\partial v_0^*}{\partial\eta}(\eta u_1^* - v_1^*) - \frac{\partial v_0^*}{\partial\zeta}(\zeta u_1^* - w_1^*) - (\eta u_0^* - v_0^*)\frac{\partial v_1^*}{\partial\eta} - (\zeta u_0^* - w_0^*)\frac{\partial v_1^*}{\partial\zeta} = \nabla^2 v_1^* - \frac{\partial p_1^*}{\partial\eta}$$

$$-u_0^* w_1^* - w_0^* u_1^* - \frac{\partial w_0^*}{\partial\eta}(\eta u_1^* - v_1^*) - \frac{\partial w_0^*}{\partial\zeta}(\zeta u_1^* - w_1^*) - (\eta u_0^* - v_0^*)\frac{\partial w_1^*}{\partial\eta} - (\zeta u_0^* - w_0^*)\frac{\partial w_1^*}{\partial\zeta} = \nabla^2 w_1^* - \frac{\partial p_1}{\partial\zeta}$$

$$-\eta\frac{\partial u_1^*}{\partial\eta} - \zeta\frac{\partial u_1^*}{\partial\zeta} + \frac{\partial v_1^*}{\partial\eta} + \frac{\partial w_1^*}{\partial\zeta} = 0 \qquad (11)$$

and the boundary conditions from Eq. (2) are

$$\begin{aligned}
&\eta = 0; \quad u_1^* = 0, \quad v_1^* = v_c^*(\zeta), \quad w_1^* = 0\\
&\zeta = 0; \quad u_1^* = 0, \quad v_1^* = 0, \quad w_1^* = v_c^*(\eta)\\
&\eta \to \infty; \quad u_1^* = \bar{u}_1(\zeta), \quad v_1^* = \bar{w}_1(\zeta), \quad w_1^* = \bar{v}_1(\zeta)\\
&\zeta \to \infty; \quad u_1^* = \bar{u}_1(\eta), \quad v_1^* = \bar{v}_1(\eta), \quad w_1^* = \bar{w}_1(\eta)
\end{aligned} \qquad (12)$$

Knowing u_0^*, v_0^*, w_0^* and choosing $v_c^*(t) = e^{-\alpha t} - 1$, $\alpha = 10$, Eqs. (11) and (12) were solved numerically. The results are now used in the approximation $u^* \approx u_0^* + \epsilon u_1^*$, which is shown in Fig. 2 for the symmetry plane of the corner in the cases $\epsilon = 0$, 0.1, and 0.3.

Large ϵ Solution

The treatment for large ϵ is strictly analogous to the case for large σ dealt with in Ref. 1. Whenever the variable σ appears in the latter analysis, it should be replaced in the present case by ϵ and the solution will follow as before. The only difference is in the meaning of the asymptotic state reached for large σ or ϵ. As $\sigma \to \infty$ the flow approaches an asymptotic state in which the physical velocity vector becomes independent of σ. For large ϵ, on the other hand, the flow reaches an asymptotic state in which the forms taken by the physical velocity components are universal but parametrically dependent on ϵ. See, for example, Eqs. (8) and (9) above. Results for $u^*(\eta,\eta)$ with $\epsilon = 2$, 4, and 10 are shown in Fig. 2.

Approximate Solution for Arbitrary ϵ

For ϵ values intermediate to small and large, we follow Ref. 1 in its treatment for arbitrary σ and adapt the approximate method of Carrier[9] to the present problem. We assume that

$$u^*(\eta,\zeta) = g_{\eta\zeta}, \quad v^*(\eta,\zeta) = \eta g_{\eta\zeta} - g_\zeta, \quad w^*(\eta,\zeta) = \zeta g_{\eta\zeta} - g_\eta$$

where a subscript η or ζ denotes differentiation with respect to that variable. The assumed forms satisfy the continuity equation (1d) identically. Cross flow Eqs. (1b) and (1c) are neglected and Eq. (1a) becomes

$$g_{\eta\eta\eta\zeta} + g_{\eta\zeta\zeta\zeta} + g_\eta g_{\eta\zeta\zeta} + g_\zeta g_{\eta\eta\zeta} = 0 \qquad (13)$$

subject to the boundary conditions

$$\eta=0;\quad g_{\eta\zeta}=0,\quad g_\zeta=-\epsilon v_c^*(\zeta),\quad g_\eta=0$$
$$\zeta=0;\quad g_{\eta\zeta}=0,\quad g_\zeta=0,\quad g_\eta=-\epsilon v_c^*(\eta)$$
$$\eta\to\infty;\quad g_{\eta\zeta}=\bar u(\zeta;\epsilon)$$
$$\zeta\to\infty;\quad g_{\eta\zeta}=\bar u(\eta;\epsilon) \tag{14}$$

Equations (13) and (14) were solved numerically for several values of ϵ, assuming that $v_c^*(t)=e^{-10t}-1$. The results are illustrated in Fig. 2.

The conclusions drawn from the results of the approximate solution are precisely the same as for the case of arbitrary σ (Ref. 1) and for essentially the same reasons. The approximate solution gives a poor representation of the flow at small ϵ, but virtually coincides with the exact solution when ϵ is sufficiently large (Fig. 2). It is reasonable to infer that this improvement in accuracy from poor to excellent is continuous in ϵ and, therefore, that the approximate solution may have some utility for ϵ values beyond the range of the small ϵ exact solution.

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Stability of Time Finite Elements

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Introduction

THE finite element method was originally developed for the approximate solution of boundary value problems. It was a natural step to attempt to develop similar methods for initial value problems. Some of these efforts have been summarized by Zienkiewicz[1] and Oden.[2] For the equations of motion of the dynamical systems to be satisfied, it is necessary to provide specifications for the displacements and velocities; otherwise, the initial conditions cannot be selected simply. The lowest-order interpolation set is thus that of the piecewise third-order Hermitian interpolation polynomials. Indeed, most of the published time finite element algorithms for the dynamic problems make use of these interpolation polynomials (see, e.g., Refs. 2-7).

The purpose of this Note is to prove that the straightforward application of the Ritz or Galerkin procedures with these approximation functions will result in an unstable algorithm. Unfortunately, this fact does not appear to be too widely known. A modification of the conventional procedures that provides a consistent stable algorithm has been proposed by the authors.[8]

For simplicity, we consider only the linear, one degree-of-freedom, problem

$$m\ddot u+c\dot u+ku=f,\qquad u(0)=u_0,\qquad \dot u(0)=\dot u_0 \tag{1}$$

where m,c,k are the mass, damping, and stiffness parameters, respectively, and u and f the time-dependent displacement and load. Time derivatives are denoted by a dot over the quantity.

The Conventional Time Finite Element

As suggested in Refs. 7-9, let us start with Hamilton's law of varying action, which for the system given in Eq. (1) results in the form:

$$\int_{t_0}^{t_f}(\delta\dot u m\dot u-\delta u c\dot u-\delta u k u+\delta u f)\,\mathrm{d}t-\delta u m\dot u\Big|_{t_0}^{t_f}=0 \tag{2}$$

or from the more constrained version[9,10] of Hamilton's principle as suggested in Refs. 2-5,

$$\int_{t_0}^{t_f}(\delta\dot u m\dot u-\delta u c\dot u-\delta u k u+\delta u f)\,\mathrm{d}t=0;\quad \delta u(t_0)=\delta u(t_f)=0 \tag{3}$$

In accordance with the finite element technique, the integral expressions of Eqs. (2) and (3) can be discretized and expressed as a sum over time finite elements.[8] Prescribing the same interpolation procedures as in Refs. 2-7, and summing the contribution of all the elements yields the following expression for the integral and the summation in Eqs. (2) and (3), respectively:

$$U^T[K'U-F]=0 \tag{4}$$

$$U^T[KU-F]=0 \tag{5}$$

where K' and K are $(2n+2)\times(2n+2)$ matrices that differ only in the presence or absence of two boundary terms, and $U^T=[u_0,\dot u_0,u_1,\dot u_1,\ldots,u_n,\dot u_n]$.

Assuming regular element size at Δt, a typical row (or rather pairs of rows, since there are two unknowns u_j and $\dot u_j$ at the mesh point $t_j=j\Delta t$) of the systems of Eqs. (4) and (5) is[11]—no matter how one choses to impose the initial conditions,[3,4,7]

$$\begin{aligned}&\left(\frac{6}{5}m-\frac{\Delta t}{2}c+\frac{9\Delta t^2}{70}k\right)u_{j-1}+\left(\frac{1}{10}m-\frac{\Delta t}{10}c\right.\\&\left.+\frac{13\Delta t^2}{420}k\right)\Delta t\dot u_{j-1}+\left(-\frac{12}{5}m+\frac{26\Delta t^2}{35}k\right)u_j\\&+\frac{\Delta t^2}{5}c\dot u_j+\left(\frac{6}{5}m+\frac{\Delta t}{2}c+\frac{9\Delta t^2}{70}k\right)u_{j+1}\\&+\left(-\frac{1}{10}m-\frac{\Delta t}{10}c-\frac{13\Delta t^2}{420}k\right)\Delta t\dot u_{j+1}\\&=\frac{\Delta t^2}{420}(54f_{j-1}+13\Delta t f_{j-1}+312f_j+54f_{j+1}-13\Delta t\dot f_{j+1})\end{aligned} \tag{6}$$

Received Sept. 12, 1983.

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